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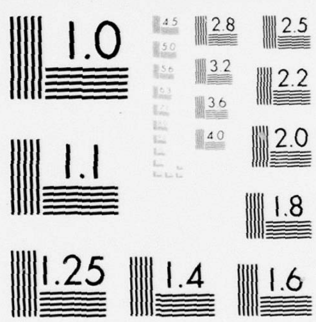

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Report No. 102

LIE-THEORY FOR A ONE-DIMENSIONAL LAMINATED COMPOSITE AND THE PROBLEM OF HOMOGENIZATION*

by

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Abstract. The paper deals with the development of a continuum theory for time-harmonic longitudinal waves propagating in a one-dimensional laminated composite with periodic structure. The development is based on the method of Lie-series and a hierarchy of improved approximate theories are obtained for wavelengths longer than the typical dimensions of the composite microstructure. Effective material properties are also defined and the dispersion spectrum compared with the exact theory, over one Brillouin zone. It is also shown how the effective properties of the laminated, periodic composite can also be obtained by homogenization via a multiple scale expansion.

Introduction

When studying the behavior of an inhomogeneous material with material properties which vary with position, it often proves convenient to replace it by a homogeneous material, whose properties approximate that of the original material, in some suitable manner. The properties of this 'equivalent' homogeneous material are then called the effective or bulk properties of the inhomogeneous material. This idea is quite old and can be traced back to Poisson, Maxwell and Rayleigh. An extensive survey and references on this subject can be found in the article by Babuška [1]*.

Recently, with increasing use of composite materials with laminated structure, and the increasing complexity of the problems generated by their study, the question of developing 'approximate' theories using effective properties has become increasingly important. Earlier work in this area can be classified in three categories, i) effective stiffness theory, ii) mixture theory, and iii) low frequency long wavelength theory. Representative of these three categories are the papers by i) Sun, Achenbach, and Herrmann [2], ii) Bedford and Stern [3], and iii) Kohn [4], respectively. In this paper we outline in a compact and formally exact form two such theories, one using the method of Lie-series [5] and the other using homogenization via multiple scale expansion. We show that the two methods of developing the approximate theory for elastic waves in laminated composites lead to equivalent results. Furthermore, in the limit of low frequency and long wavelength these equations are in agreement with the equations obtained earlier by Kohn. Therefore

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* Numbers in brackets designate References at end of paper.

these theories, though obtained by using three entirely different mathematical procedures, may be called long wavelength approximation theories. The accuracy of these approximate equations can be improved by increasing the order of the differential equation. Our results show that the approximate dispersion spectrum tends monotonically towards the exact spectrum by increasing the order of approximation and the convergence is from below. We have shown that in the case of elastic waves in one dimensional laminated composite, a linear differential operator of the form $(\partial_t^2 - \sum_{n=1}^{4} C_n^2 \partial_y^{2n})$, reproduces the exact spectrum in the first Brillouin zone to a high degree of accuracy.

The method of Lie-Series as used in this paper has several advantages over other methods used in the past in the development of approximate theories. First convergence can be justified and there exists a large literature on this subject. Second, using Lie's exponential mapping and commutation theorems one gets the exact spectrum from the Lie-form of differential equations. The correspondence between a spectral function and its differential equation, in conjunction with the implicit function theorem, then helps us to develop a hierarchy of improved approximate theories.

Formulation using Lie-Series

We consider the one dimensional elastic wave equation

$$\partial\sigma/\partial y - \rho\partial^2 U/\partial t^2 = 0, \qquad 0 < y < \infty, \quad t > 0 \tag{1}$$

$$\sigma = \eta\partial U/\partial y , \tag{2}$$

where $\sigma \epsilon C^2$ is the stress, $U \epsilon C^2$ is the longitudinal displacement, ρ the mass density, η the stiffness, y the longitudinal distance parameter in the undeformed configuration, and t the parameter of time. The one-

dimensional medium is supposed to be laminated with period $2\Delta \equiv 2(h_1 + h_2)$. The unit cell is the union of two materials; part of the cell with thickness $2h_1$ has material properties (ρ_1, η_1), and the remainder of the cell with thickness $2h_2$ has material properties (ρ_2, η_2), as shown in Fig. 1. The union of an infinity of such unit cells gives us the periodic laminated structure with period 2Δ with respect to y. The one dimensional longitudinal wave equation applies to each part of the cell, though different parts of the cell have different material properties.

The continuity of displacement and stress at the interface A requires that

$$\begin{aligned} U_1^k(Y_1^k + h_1, t) &= U_2^k(Y_2^k - h_2, t), \qquad t \geq 0 \\ \sigma_1^k(Y_1^k + h_1, t) &= \sigma_2^k(Y_2^k - h_2, t), \qquad t \geq 0 \end{aligned} \tag{3}$$

and similarly at the interface B, the continuity conditions are

$$\begin{aligned} U_2^k(Y_2^k + h_2, t) &= U_1^{k+1}(Y_1^{k+1} - h_1, t), \qquad t \geq 0 \\ \sigma_2^k(Y_2^k + h_2, t) &= \sigma_1^{k+1}(Y_1^{k+1} - h_1, t), \qquad t \geq 0 \end{aligned} \tag{4}$$

where the superscript k indexes the cell. The distance from the origin of the mass center of layer $\alpha(=1 \text{ or } 2)$ of the k-th cell, $1 \leq k < \infty$, is denoted by Y_α^k.

Equations (1) and (2) when combined can be written as

$$\left(\frac{\partial}{\partial y} - \beta \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y} + \beta \frac{\partial}{\partial t}\right) U(y,t) = 0 \ , \quad 0 \leq y < \infty, \quad t > 0 \tag{5}$$

where $\beta^2 = \rho/\eta$ is a constant and represents the reciprocal of the square of wave speed. Since β is a constant, the two differential operators commute and the general solution can be written as the sum of the solutions of each.

Consider the initial boundary value problem

$$\begin{aligned} &\frac{\partial U}{\partial y} - \beta \frac{\partial U}{\partial t} = 0, \qquad 0 < y < \infty, \qquad t > 0 \\ &U(y,t)\Big|_{y=o} = \bar{U}_o(Ct), \qquad\qquad t > 0 \end{aligned} \tag{6}$$

where $C \equiv 1/\beta$ and $\bar{U}_o \varepsilon C^{\omega}$. By the exponential mapping theorem the solution of this simple problem is given by the Lie-series

$$U(y,t) = e^{yD}\, \bar{U}_o(Ct) \ , \tag{7}$$

where the operator D is given by

$$D \equiv \beta \frac{\partial}{\partial t} \ . \tag{8}$$

Similarly the solution of the second initial boundary value problem

$$\begin{aligned} &\frac{\partial U}{\partial y} + \beta \frac{\partial U}{\partial t} = 0 \ , \qquad 0 < y < \infty, \quad t > 0 \\ &U(y,t)\Big|_{y=o} = \bar{U}_1(Ct), \qquad\qquad t > 0 \end{aligned} \tag{9}$$

where $\bar{U}_1 \varepsilon C^{\omega}$, is given by the Lie-series

$$U(y,t) = e^{-yD}\, \bar{U}_1(Ct) \ . \tag{10}$$

Thus the general solution of the one dimensional wave equation (5), is given in terms of the Lie-series by

$$U(y,t) = e^{yD}\, \bar{U}_o(Ct) + e^{-yD}\, \bar{U}_1(Ct). \tag{11}$$

Using the commutation theorem of Lie operators [5], this solution can be written as

$$\begin{aligned} U(y,t) &= \bar{U}_o(e^{yD}\, Ct) + \bar{U}_1(e^{-yD}\, Ct), \\ &= \bar{U}_o(Ct + y) + \bar{U}_1(Ct - y) \ , \end{aligned} \tag{12}$$

which is the standard D'Alembert solution of wave equation, and represents two waves propagating with speed C without change of form in the positive and negative directions.

The solution (11) can further be rewritten as

$$U(y,t) = (\cosh yD)U_o(Ct) + (\sinh yD)U_1(Ct) \ , \tag{13}$$

where $U_o = (\bar{U}_o + \bar{U}_1)/2$, $U_1 = (\bar{U}_o - \bar{U}_1)/2$ and are suitable analytic functions of the parameter t . When the operator D and the functions U_o and U_1 are all analytic, there exists a positive number $\tilde{Y}$, such that the Lie-series (13) is uniformly convergent for $|y| < \tilde{Y}$, and therefore termwise differentiation is admissible. Then from Eqn. (2) and termwise differentiation of Eqn. (13), we find that the stress is given by

$$\sigma(y,t) = \eta D[(\sinh yD)U_o(Ct) + (\cosh yD)U_1(Ct)] \ . \tag{14}$$

From Eqns. (13) and (14) we also find that

$$U(o,t) = U_o(Ct) \ , \tag{15}$$

$$\sigma(o,t) = \eta DU_1(Ct) = \eta \frac{\partial}{\partial y} U(o,t) \equiv \sigma_o(Ct) \ ,$$

where $U_o(Ct)$ and $\sigma_o(Ct)$ are both analytic functions of t , and represent the values of displacement and stress at $y = 0$, respectively. Therefore in terms of U_o and σ_o the Lie-series representation of velocity and stress can be written as

$$\begin{aligned} \dot{U}(y,t) &= (\cosh yD)\dot{U}_o(Ct) + \frac{1}{C\rho}(\sinh yD)\sigma_o(Ct) \ , \\ \sigma(y,t) &= (\cosh yD)\sigma_o(Ct) + C\rho(\sinh yD)\dot{U}_o(Ct) \ , \end{aligned} \tag{16}$$

where dot indicates differentiation with respect to t, and use is made of the fact that the operator D commutes with the operators coshyD and sinhyD when U_o and $U_1 \varepsilon C^{\omega}$. We thus see that if the C^{ω} functions $U_o(Ct)$ and $\sigma_o(Ct)$ are the values of displacement and stress at some midpoint $y = 0$, then in terms of Lie-series and exponential mapping theorem, the values of velocity and stress at some typical point (y,t) are given by

$$\begin{bmatrix} \dot{U}(y,t) \\ \sigma(y,t) \end{bmatrix} = \begin{bmatrix} \cosh yD & \frac{1}{C\rho} \sinh yD \\ C\rho \sinh yD & \cosh yD \end{bmatrix} \begin{bmatrix} \dot{U}_o(Ct) \\ \sigma_o(Ct) \end{bmatrix} , \qquad (17)$$

$$0 < y < \infty , \qquad t > 0$$

where coshyD and sinhyD are Lie operators with the definitions

$$\cosh yD \equiv \sum_{o}^{\infty} \frac{1}{(2n)!} (yD)^{2n} , \quad \sinh yD \equiv \sum_{1}^{\infty} \frac{1}{(2n-1)!} (yD)^{2n-1} , \qquad (18)$$

$$\cosh^2 yD - \sinh^2 yD = 1 .$$

We may remark in passing that the matrix Eqn. (17) has an inverse, since the determinant $\cosh^2 yD - \sinh^2 yD = 1$.

Continuity Conditions and Differential Equation

At the interfaces A and B we must require the continuity of stress and longitudinal velocity. Thus from Eqns. (3) and (4) the continuity conditions are

$$
\left.\begin{aligned}
\dot{U}_1^k(Y_1^k + h_1, t) &= \dot{U}_2^k(Y_2^k - h_2, t) , \\
\dot{U}_2^k(Y_2^k + h_2, t) &= \dot{U}_1^{k+1}(Y_1^{k+1} - h_1, t) , \\
\sigma_1^k(Y_1^k + h_1, t) &= \sigma_2^k(Y_2^k - h_2, t) , \\
\sigma_2^k(Y_2^k + h_2, t) &= \sigma_1^{k+1}(Y_1^{k+1} - h_1, t) .
\end{aligned}\right\} \quad t > 0 \qquad (19)
$$

In terms of midpoint longitudinal velocity $\dot{U}^k_{o,\alpha}(Ct)$ and midpoint stress $\sigma^k_{o,\alpha}(Ct)$, $\alpha = 1,2,$ and the general cell solution (17), the continuity conditions at the interfaces A and B require that

$$
\begin{aligned}
C_{h_1} \dot{U}^k_{o,1} + \frac{1}{C_1\rho_1} S_{h_1} \sigma^k_{o,1} &= C_{h_2} \dot{U}^k_{o,2} - \frac{1}{C_2\rho_2} S_{h_2} \sigma^k_{o,2} , \\
C_{h_2} \dot{U}^k_{o,2} + \frac{1}{C_2\rho_2} S_{h_2} \sigma^k_{o,2} &= C_{h_1} \dot{U}^{k+1}_{o,1} - \frac{1}{C_1\rho_1} S_{h_1} \sigma^{k+1}_{o,1} , \\
C_{h_1} \sigma^k_{o,1} + C_1\rho_1 S_{h_1} \dot{U}^k_{o,1} &= C_{h_2} \sigma^k_{o,2} - C_2\rho_2 S_{h_2} \dot{U}^k_{o,2} , \\
C_{h_2} \sigma^k_{o,2} + C_2\rho_2 S_{h_2} \dot{U}^k_{o,2} &= C_{h_1} \sigma^{k+1}_{o,1} - C_1\rho_1 S_{h_1} \dot{U}^{k+1}_{o,1} ,
\end{aligned} \qquad (20)
$$

where

$$
\left.\begin{aligned}
C_{h_\alpha} &\equiv \cosh h_\alpha \beta_\alpha \frac{\partial}{\partial t} , \\
S_{h_\alpha} &\equiv \sinh h_\alpha \beta_\alpha \frac{\partial}{\partial t} ,
\end{aligned}\right\} \quad \alpha = 1 \text{ or } 2 \qquad (21)
$$

$$
\beta_\alpha \equiv 1/C_\alpha = (\rho_\alpha/\eta_\alpha)^{1/2} ,
$$

and $\alpha=1,(2)$ for portion of the cell with material properties, η_1,ρ_1, (η_2,ρ_2) and layer thickness $2h_1,(2h_2)$, respectively. By first adding and then subtracting the first and subsequently the second pair of equations we obtain the following equivalent set of equations

$$
\begin{aligned}
C_{h_1}(\dot{U}^{k+1}_{o,1} + \dot{U}^{k}_{o,1}) - \frac{1}{C_1\rho_1} S_{h_1}(\sigma^{k+1}_{o,1} - \sigma^{k}_{o,1}) &= 2\, C_{h_2}\, \dot{U}^{k}_{o,2} \,, \\
C_{h_1}(\dot{U}^{k+1}_{o,1} - \dot{U}^{k}_{o,1}) - \frac{1}{C_1\rho_1} S_{h_1}(\sigma^{k+1}_{o,1} + \sigma^{k}_{o,1}) &= \frac{2}{C_2\rho_2} S_{h_2}\sigma^{k}_{o,2} \,, \\
S_{h_1}(\dot{U}^{k+1}_{o,1} - \dot{U}^{k}_{o,1}) - \frac{1}{C_1\rho_1} C_{h_1}(\sigma^{k+1}_{o,1} + \sigma^{k}_{o,1}) &= -\frac{2}{C_1\rho_1} C_{h_2}\, \sigma^{k}_{o,2} \,, \\
S_{h_1}(\dot{U}^{k+1}_{o,1} + \dot{U}^{k}_{o,1}) - \frac{1}{C_1\rho_1} C_{h_1}(\sigma^{k+1}_{o,1} - \sigma^{k}_{o,1}) &= -2\,\frac{C_2\rho_2}{C_1\rho_1} S_{h_2}\dot{U}^{k}_{o,2} \,.
\end{aligned}
\tag{22}
$$

We shall now postulate the existence of analytic functions $U_\alpha(y,t)$ and $\sigma_\alpha(y,t)$, with respective radii of convergence greater than the half-period Δ, and with the important property

$$
U_\alpha(Y^k_\alpha,t) = U^k_{o,\alpha}(t) \; ; \qquad \sigma_\alpha(Y^k_{\alpha,t}) = \sigma^k_{o,1}(t) \,. \tag{23}
$$

Due to the strong smoothness conditions imposed on these functions it is possible to write

$$
\begin{aligned}
U^k_{o,1}(t) &= U_1(Y^k_1,t) = U_1(Y^k_2 - \Delta,t) = \lim_{y\to Y^k_2} e^{-\Delta\partial_y} U_1(y,t) \,, \\
U^{k+1}_{o,1}(t) &= U_1(Y^{k+1}_1,t) = U_1(Y^k_2 + \Delta,t) = \lim_{y\to Y^k_2} e^{+\Delta\partial_y} U_1(y,t) \,, \\
\sigma^k_{o,1}(t) &= \sigma_1(Y^k_1,t) = \sigma_1(Y^k_2 - \Delta,t) = \lim_{y\to Y^k_2} e^{-\Delta\partial_y} \sigma_1(y,t) \,, \\
\sigma^{k+1}_{o,1}(t) &= \sigma_1(Y^{k+1}_1,t) = \sigma_1(Y^k_2 + \Delta,t) = \lim_{y\to Y^k_2} e^{\Delta\partial_y} \sigma_1(y,t) \,,
\end{aligned}
\tag{24}
$$

where $\partial_y \equiv \partial/\partial y$ and use is made of Lie's commutation theorem. From Eqns. (24) and (22) it now follows that at the station Y_2^k the functions $\dot{U}_\alpha$ and σ_α satisfy the equations

$$
\begin{aligned}
&\lim_{y\to Y_2^k}\left[C_{h_1} C_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} S_{h_1} S_\Delta \sigma_1 - C_{h_2}\dot{U}_2\right] = 0 \ , \\
&\lim_{y\to Y_2^k}\left[C_{h_1} S_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} S_{h_1} C_\Delta \sigma_1 - \frac{1}{C_2\rho_2} S_{h_2}\sigma_2\right] = 0 \ , \\
&\lim_{y\to Y_2^k}\left[S_{h_1} S_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} C_{h_1} C_\Delta \sigma_1 + \frac{1}{C_1\rho_1} C_{h_2}\sigma_2\right] = 0 \ , \\
&\lim_{y\to Y_2^k}\left[S_{h_1} C_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} C_{h_1} S_\Delta \sigma_1 + \frac{C_2\rho_2}{C_1\rho_1} S_{h_2}\dot{U}_2\right] = 0 \ ,
\end{aligned}
\tag{25}
$$

where

$$C_\Delta \equiv \cosh\Delta\partial_y \qquad ; \qquad S_\Delta \equiv \sinh\Delta\partial_y \ . \tag{26}$$

In addition we assume that among the entire class of C^ω functions fitting the midpoint solutions there exists a particular set which satisfy Eqns. (25) for all values of y , rather than at the station Y_2^k only. This special set of functions $\{\dot{U}_\alpha, \sigma_\alpha\}$ could then be characterized as a solution to the set of four, infinite order, linear partial differential equations

$$
\begin{aligned}
&C_{h_1} C_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} S_{h_1} S_\Delta \sigma_1 = C_{h_2}\dot{U}_2 \ , \\
&C_{h_1} S_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} S_{h_1} C_\Delta \sigma_1 = \frac{1}{C_2\rho_2} S_{h_2}\sigma_2 \ , \\
&S_{h_1} S_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} C_{h_1} C_\Delta \sigma_1 = -\frac{1}{C_1\rho_1} C_{h_2}\sigma_2 \ , \\
&S_{h_1} C_\Delta \dot{U}_1 - \frac{1}{C_1\rho_1} C_{h_1} S_\Delta \sigma_1 = -\frac{C_2\rho_2}{C_1\rho_1} S_{h_2}\dot{U}_2 \ .
\end{aligned}
\tag{27}
$$

Now one can express $\dot{U}_1$, and σ_1 in terms of $\dot{U}_2$ and σ_2 from the first pair of Eqns. $(27)_1$, $(27)_2$ and likewise from the second pair of Eqns. $(27)_3$ and $(27)_4$. Thus from Eqns. (27) we can easily obtain another set of four equations

$$
\begin{aligned}
C_{h_2}C_\Delta \dot{U}_2 - \frac{1}{C_2\rho_2} S_{h_2}S_\Delta\sigma_2 &= C_{h_1}\dot{U}_1 , \\
C_{h_2}S_\Delta \dot{U}_2 - \frac{1}{C_2\rho_2} S_{h_2}C_\Delta\sigma_2 &= \frac{1}{C_1\rho_1} S_{h_1}\sigma_1 , \\
S_{h_2}S_\Delta \dot{U}_2 - \frac{1}{C_2\rho_2} C_{h_2}C_\Delta\sigma_2 &= -\frac{1}{C_2\rho_2} C_{h_1}\sigma_1 , \\
S_{h_2}C_\Delta \dot{U}_2 - \frac{1}{C_2\rho_2} C_{h_2}S_\Delta\sigma_2 &= -\frac{C_1\rho_1}{C_2\rho_2} S_{h_1}\dot{U}_1 .
\end{aligned}
\tag{28}
$$

It may be remarked that the two sets of equations may also be obtained by expanding the analytic functions $\{\dot{U}_\alpha,\sigma_\alpha\}$ about the stations Y_2^k and Y_1^{k+1} and using the continuity conditions at the three interfaces $(Y_1^k + h_1)$, $(Y_2^k + h_2)$, and $(Y_1^{k+1} + h_1)$, respectively.

To obtain uncoupled systems of equations, we eliminate $\dot{U}_2$ and σ_2 from the set of Eqns. (27) and, $\dot{U}_1$ and σ_1 from the set of Eqns. (28). This elimination process thus leads us to two pairs of uncoupled equations

$$
\begin{aligned}
(S_{h_1}C_{h_2} + \theta C_{h_1}S_{h_2})C_\Delta\dot{U}_1(y,t) - \frac{1}{C_1\rho_1}(C_{h_1}C_{h_2} + \theta S_{h_1}S_{h_2})S_\Delta\sigma_1(y,t) &= 0, \\
(S_{h_1}S_{h_2} + \theta C_{h_1}C_{h_2})S_\Delta\dot{U}_1(y,t) - \frac{1}{C_1\rho_1}(C_{h_1}S_{h_2} + \theta S_{h_1}C_{h_2})C_\Delta\sigma_1(y,t) &= 0, \\
(S_{h_1}C_{h_2} + \theta C_{h_1}S_{h_2})C_\Delta\dot{U}_2(y,t) - \frac{1}{C_2\rho_2}(S_{h_1}S_{h_2} + \theta C_{h_1}C_{h_2})S_\Delta\sigma_2(y,t) &= 0, \\
(C_{h_1}C_{h_2} + \theta S_{h_1}S_{h_2})S_\Delta\dot{U}_2(y,t) - \frac{1}{C_2\rho_2}(C_{h_1}S_{h_2} + \theta S_{h_1}C_{h_2})C_\Delta\sigma_2(y,t) &= 0,
\end{aligned}
\tag{29}
$$

where $\theta \equiv C_2\rho_2/C_1\rho_1$.

Let $\overset{*}{D}_1$ be the determinant of the coefficients of $\dot{U}_1$ and σ_1, and $\overset{*}{D}_2$ the determinant of the coefficients of $\dot{U}_2$ and σ_2, in the system of homogeneous Eqns. (29). It is easy to verify that $\overset{*}{D}_1 = \overset{*}{D}_2$ (= say $\overset{*}{D}$), and therefore each of the analytic functions $\dot{U}_1, \sigma_1, \dot{U}_2, \sigma_2 (=\Phi)$, where Φ is a collective symbol, satisfies a differential equation of the form $\overset{*}{D}\Phi = 0$. Explicitly, the infinite order, linear, partial differential equation takes the form

$$\left[C_{2\Delta} - \frac{1}{2}(\theta + 1/\theta)\, S_{2h_1} S_{2h_2} - C_{2h_1} C_{2h_2}\right]\Phi = 0 \,. \tag{30}$$

By slight rearrangement this equation can be put in the more useful form

$$\left[C_{2\Delta} - \frac{1}{4}\left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)^2 C_{2(h_1+h_2)} + \frac{1}{4}\left(\sqrt{\theta} - \frac{1}{\sqrt{\theta}}\right)^2 C_{2(h_1-h_2)}\right]\Phi = 0 \,. \tag{31}$$

We have thus replaced our infinite set of governing equations (1) and (2) with a set of four partial differential equations. From these equations a series of approximate equations of increasing order can be obtained by retaining only a finite number of terms in the symbolic operators

$$\begin{aligned} C_{2\Delta} &\equiv \cosh 2\Delta\frac{\partial}{\partial y} = \sum_0^\infty \frac{(2\Delta)^{2n}}{(2n)!}\left(\frac{\partial}{\partial y}\right)^{2n} \,, \\ C_{2(h_1\pm h_2)} &\equiv \cosh 2(h_1\beta_1 \pm h_2\beta_2)\frac{\partial}{\partial t} = \sum_0^\infty \frac{[2(h_1\beta_1 \pm h_2\beta_2)]^{2n}}{(2n)!}\left(\frac{\partial}{\partial t}\right)^{2n} \,. \end{aligned} \tag{32}$$

For developing the higher order equations we will use Eqn. (31), because all the symbolic operators in this equation are linear. However, we will have occasion to use Eqn. (30) when we discuss the spectral function

corresponding to this operator equation.

The lowest order approximation to Eqn. (31) can easily be obtained if we retain the first two terms in the expansion of the symbolic operators entering in this equation. Thus, after simplification we obtain

$$\left[\frac{\partial^2}{\partial y^2} - \left(\frac{h_1\rho_1 + h_2\rho_2}{h_1 + h_2}\right)\left(\frac{h_1/\eta_1 + h_2/\eta_2}{h_1 + h_2}\right)\frac{\partial^2}{\partial t^2}\right]\Phi = 0\ . \qquad (33)$$

$$0 < y < \infty,\ t > 0$$

This is a one-dimensional wave equation and from this equation, the effective longitudinal phase velocity, density and modulus can easily be read. Thus we find

i. effective zero-order density

$$\overset{(o)}{\rho} = (h_1\rho_1 + h_2\rho_2)/(h_1 + h_2)\ ,$$

ii. effective zero-order modulus

$$\overset{(o)}{\eta} = (h_1 + h_2)/(h_1/\eta_1 + h_2/\eta_2)\ , \qquad (34)$$

iii. effective zero-order phase velocity

$$\overset{(o)}{C}{}^2 = \left[(\rho_1/\rho_2 + h_2/h_1)(\eta_2/\eta_1 + h_2/h_1)/[C_2(1 + h_2/h_1)]^2\right]^{-1} = \overset{(o)}{\eta}/\overset{(o)}{\rho}\ .$$

The zero-order phase velocity is the effective celerity of the longitudinal wave motion, with wavelengths much longer than the thickness of the laminations. We may therefore call it the long wavelength sound velocity of the laminated composite. This result is well known and was obtained by Rytov [6], and simultaneously by White and Angona [7]. The methods used by these authors are different from the one developed in this paper, and in the second part of this paper we will show how

the zero-order effective properties of a laminated medium can also be obtained by using a multiple scale expansion of the displacement field.

The lowest order approximation yields a non-dispersive wave equation. The initial value problem corresponding to this equation requires that the Cauchy data $\Phi(y,o)$, $\dot{\Phi}(y,o)$ be prescribed at every point on the support $0 < y < \infty$, $t=0$.

Before developing higher order equations, we first introduce non-dimensional variables ζ and τ, related to our original variables y and t, respectively, as follows

$$T : \quad y \rightarrow \Delta\zeta \ , \qquad t \rightarrow (\Delta/C_1)\tau \ , \tag{35}$$

where we have used C_1 as the reference speed of the lamina with material properties η_1, ρ_1. Furthermore, we introduce material, geometric and kinematical properties in terms of the ratio $\tilde{\rho}$, $\tilde{\eta}$, $\tilde{h}$ and $\tilde{c}$, where

$$\tilde{\rho} \equiv \rho_1/\rho_2 \ , \quad \tilde{\eta} \equiv \eta_1/\eta_2 \ , \quad \tilde{h} \equiv h_1/h_2 \ , \quad \tilde{c} \equiv C_1/C_2 \ . \tag{36}$$

In terms of the new variables ζ and τ and the parameters $\tilde{\rho}$, $\tilde{\eta}$, $\tilde{h}$ and $\tilde{c}$, the linear, infinite order, partial differential equation (31), takes the non-dimensional form

$$\begin{aligned}\Bigg[\cosh 2\frac{\partial}{\partial\zeta} - \frac{1}{4}\left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)^2 \cosh \frac{2(1+\tilde{c}/\tilde{h})}{1+1/\tilde{h}}\frac{\partial}{\partial\tau} \\ + \frac{1}{4}\left(\sqrt{\theta} - \frac{1}{\sqrt{\theta}}\right)^2 \cosh \frac{2(1-\tilde{c}/\tilde{h})}{1+1/\tilde{h}}\frac{\partial}{\partial\tau}\Bigg] \Phi(\zeta,\tau) = 0 \ ,\end{aligned} \tag{37}$$

for $0 < \zeta < \infty$, $\tau > 0$ and where $\theta \equiv 1/(\tilde{c}\tilde{\rho}) = \tilde{c}/\tilde{\eta}$.

Approximate differential equations of increasingly higher order are obtained from Eqn. (37) by systematic use of the expansions of the symbolic operators entering in this equation. Thus, partial differential equations of fourth and sixth order are

i. fourth order approximation

$$\left[\left(\frac{\partial^2}{\partial\zeta^2} - \alpha^2 \frac{\partial^2}{\partial\tau^2}\right) + \frac{2^3}{4!}\left(\frac{\partial^4}{\partial\zeta^4} - \beta^4 \frac{\partial^4}{\partial\tau^4}\right)\right] \Phi = 0 \ , \tag{38$_i$}$$

ii. sixth order approximation

$$\left[\left(\frac{\partial^2}{\partial\zeta^2} - \alpha^2 \frac{\partial^2}{\partial\tau^2}\right) + \frac{2^3}{4!}\left(\frac{\partial^4}{\partial\zeta^4} - \beta^4 \frac{\partial^4}{\partial\tau^4}\right) + \frac{2^5}{6!}\left(\frac{\partial^6}{\partial\zeta^6} - \gamma^6 \frac{\partial^6}{\partial\tau^6}\right)\right]\Phi = 0, \tag{38$_{ii}$}$$

where

$$\alpha^2 = \frac{(\tilde{\rho} + 1/\tilde{h})(1/\tilde{\eta} + 1/\tilde{h})}{(1 + 1/\tilde{h})^2}\,\tilde{c}^2 \ ,$$

$$\beta^4 = \frac{\left[1 + \frac{2}{\tilde{\rho}\tilde{h}} + \left(\frac{\tilde{c}}{\tilde{h}}\right)^2\right]\left[\frac{1}{\tilde{c}^2} + \frac{2\tilde{\rho}}{\tilde{h}} + \frac{1}{\tilde{h}^2}\right]}{(1 + 1/\tilde{h})^4}\,\tilde{c}^2 \tag{39}$$

$$\gamma^6 = \frac{\left[\tilde{\rho} + \frac{3}{\tilde{h}}(1 + \tilde{\eta}/\tilde{h}) + \frac{\tilde{c}^2}{\tilde{h}^3}\right]\left[\frac{1}{\tilde{\eta}} + \frac{3}{\tilde{h}}\left(1 + \frac{1}{\tilde{\rho}\tilde{h}}\right) + \frac{\tilde{c}^2}{\tilde{h}^3}\right]}{(1 + 1/\tilde{h})^6}\,\tilde{c}^2 \ .$$

These equations in their present form contain fourth and sixth order partial derivatives with respect to variable τ . However, we will show that we can replace this set of equations by an equivalent set of equations which contain only the second partial derivative with respect to τ , and thus requiring Cauchy data pertaining to $\Phi(y,o)$

and $\dot{\Phi}(y,o)$ along the support at $t=0$.

We may now remark that Eqn. (37) is exact and replaces the infinite system of Eqns. (1),(2) and interface continuity conditions (3),(4) by a single set of linear partial differential equations of infinite order. If this is exact then it must yield the same dispersion spectrum as the one we would have obtained from our original infinite system of partial differential equations with piecewise continuous and periodic coefficients. As a matter of fact, using Floquet's theory of differential equations with periodic coefficients, Rytov [6], and Lee and Yang [8], have obtained the dispersion equation for this relatively simple problem. It will therefore be instructive to compare our results with the ones given by these authors.

For dispersive waves we assume solution in the form of sinusoidal wave trains

$$\Phi = \Phi_o \exp i(q\zeta - \Omega\tau) \ , \tag{40}$$

where q is the Floquet wave number, Ω the non-dimensional frequency and Φ_o the amplitude which is arbitrary. We immediately notice that if we substitute Eqn. (40) in the non-dimensional form of Eqn. (30), then we get the same spectral function as given by Rytov, and Lee and Yang. Having established that our differential Eqn.(30) yields the exact dispersion spectrum, we now revert to Eqn. (31) whose spectral equation corresponding to sinusoidal wave trains is

$$\cos 2q - \frac{1}{4}\left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)^2 \cos \frac{2(1 + \tilde{c}/\tilde{h})}{1 + 1/\tilde{h}}\,\Omega + \frac{1}{4}\left(\sqrt{\theta} - \frac{1}{\sqrt{\theta}}\right)^2 \cos \frac{2(1 - \tilde{c}/\tilde{h})}{1 + 1/\tilde{h}}\,\Omega = 0. \tag{41}$$

As shown by Lee and Yang, the graph of this function has a banded structure, consisting of passing bands and stopping bands and the width of Brillouin zone is $\pi/2$. Our aim now is to show that from Eqn. (37) a system of approximate differential equations can be derived whose spectral graph in the first Brillouin zone agrees with the corresponding graph of the exact equation, up to any desired degree of accuracy.

Now according to a theorem by Gelfand and Levitan [9], a differential equation can be determined from its spectral function. If we establish the correspondence

$$q \leftrightarrow i \frac{\partial}{\partial \zeta} , \qquad \Omega \leftrightarrow i \frac{\partial}{\partial \tau} , \tag{42}$$

then this theorem is trivially obvious in the case of polynomial spectral function. Using this correspondence, it is also obvious that from the spectral Eqn. (41) we get the differential Eqn. (37). Hence, a sequel to this is the corollary that there is a 1:1 mapping between the $(q,\Omega)\leftrightarrow(\zeta,\tau)$ spaces. Consequently, for those problems for which the dispersion spectrum can be easily obtained by using Floquet's theory of differential equations with periodic coefficients, the corresponding operator differential equation can be immediately derived by using Gelfand - Levitan theorem. Alternatively, one may first derive the operator differential equation by using the method of Lie-series and then use this correspondence to determine the spectral function. Because of the 1:1 mapping between the operator equation and its spectral function, a power series representation in the (q,Ω)-space can be translated into a power series representation in the (ζ,τ)-space. Hence, replacing the symbolic operators by their power series representation, and

retaining terms of order Δ^{2n}, gives us an approximate differential equation of order 2n. Furthermore, the dispersion spectrum corresponding to this system of higher order equations, is just the power series representation of the exact dispersion equation up to the same order. Hence, the dispersion equations for the Eqns. (33) and (38) can be directly obtained from expansions of Eqn. (41), or conversely, from the power series representation of Eqn. (41), one can obtain a system of higher order differential Eqns. (33,38).

From Eqn. (41) we readily see that about the origin (o,o), it is an even function with respect to both variables and has a functional representation of the form $F(\Omega^2,q^2) = 0$. Therefore by the implicit function theorem, if $\partial F/\partial\Omega^2 \neq 0$ at the origin, then in the neighborhood of the origin it has the representation $\Omega^2 = f(q^2)$. Analyticity of the function f leads us to the existence of the power series representation

$$\Omega^2 = \alpha_o^2 q^2 - \beta_o^4 q^4 + \gamma_o^6 q^6 - \delta_o^8 q^8 + \dots \,. \tag{43}$$

Gelfand-Levitan theorem now asserts that corresponding to this polynomial spectral equation, the differential equation is

$$\frac{\partial^2\Phi}{\partial\tau^2} = \left(\alpha_o^2 \frac{\partial^2}{\partial\zeta^2} + \beta_o^4 \frac{\partial^4}{\partial\zeta^4} + \gamma_o^6 \frac{\partial^6}{\partial\zeta^6} + \delta_o^8 \frac{\partial^8}{\partial\zeta^8} + \dots\right)\Phi \,. \tag{44}$$

If we substitute Eqn. (44) in Eqn. (37), we get an identity. This determines the coefficients, which are

$$\alpha_o^2 = 1/\alpha^2 \ ,$$

$$\beta_o^4 = \frac{1}{3\alpha^2}(1 - \beta^4/\alpha^4) \ ,$$

$$\gamma_o^6 = \frac{8}{45\alpha^2}(1 - \gamma^6/\alpha^6) - \frac{2}{9\alpha^2}(\beta^4/\alpha^4)(1 - \beta^4/\alpha^4) \ , \tag{45}$$

$$\delta_o^8 = \frac{1}{315\alpha^2}(1 - \delta^8/\alpha^8) - \frac{2}{45}\left[(\gamma^6/\alpha^6)(1 - \beta^4/\alpha^4) + \frac{2}{3}(\beta^4/\alpha^4)(1 - \gamma^6/\alpha^6)\right]$$

$$- \frac{1}{27}(\beta^4/\alpha^4)(1 - \beta^4/\alpha^4)(1 - 5\beta^4/\alpha^4) \ .$$

Thus corresponding to Eqn. $(38)_i$, we have the equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \alpha_o^2\frac{\partial^2}{\partial\zeta^2} - \beta_o^4\frac{\partial^4}{\partial\zeta^4}\right)\Phi = 0 \ , \tag*{$(46)_i$}$$

and corresponding to Eqn. $(38)_{ii}$, we have the equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \alpha_o^2\frac{\partial^2}{\partial\zeta^2} - \beta_o^4\frac{\partial^4}{\partial\zeta^4} - \gamma_o^6\frac{\partial^6}{\partial\zeta^6}\right)\Phi = 0 \ . \tag*{$(46)_{ii}$}$$

As mentioned earlier, we now have a well posed Cauchy problem requiring only the two initial values $\Phi(\zeta,\tau_o)$ and $\dot{\Phi}(\zeta,\tau_o)$ on the support $\tau = \tau_o \geq 0$. The Eqns. $(46)_i$ and $(46)_{ii}$, with Cauchy data given on the support $\tau = \tau_o$, determine $\Phi(\zeta,\tau)$ uniquely and therefore $\ddot{\Phi}(\zeta,\tau_o)$ and $\dddot{\Phi}(\zeta,\tau_o)$ can now be determined. Knowing $\partial_\tau^n\Phi(\zeta,\tau_o)$ for $n = 0,1,2$ and 3, gives us all the necessary initial conditions, for the initial value problem governed by Eqn. $(38)_i^*$. Analogous procedure holds for Eqn. $(38)_{ii}$.

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* This follows from Cauchy-Kowalewski theorem as pointed out by Prof. R. P. Shaw.

Corresponding to Eqns. $(38/46)_i$ the dispersion equation is

$$\Omega^2 - \alpha_o^2 q^2 + \beta_o^4 q^4 = 0 \ , \tag{47}$$

which leads us to the phase velocity relation

$$\Omega/q = 1/\alpha - \frac{1}{6\alpha}(1 - \beta^4/\alpha^4)q^2 - \ldots \ . \tag{48}$$

This suggests that the zero-order phase velocity is

$$\overset{(o)}{C^2} = C_1^2/\alpha^2 \ , \tag*{$(49)_i$}$$

and the second-order correction to the long wavelength sound speed is given by

$$\overset{(2)}{C^2} = \frac{1}{36}\,(\beta^4/\alpha^4 - 1)^2\,\overset{(o)}{C^2} \ , \tag*{$(49)_{ii}$}$$

where α^2 and β^4 are given by Eqn. (39).

In the case of a homogeneous medium, $\tilde{\eta} = \tilde{h} = \tilde{\rho} = 1$ and therefore from Eqn. (39) we find that $\alpha = \beta = 1$. Consequently

$$\overset{(o)}{C^2} = C_1^2, \qquad \overset{(2)}{C} = 0 \ , \tag{50}$$

and we get the well known results for longitudinal waves in a homogeneous, isotropic, infinite one-dimensional medium.

Corresponding to Eqns. $(38/46)_{ii}$ the dispersion equation is

$$\Omega^2 - \alpha_o^2 q^2 + \beta_o^4 q^4 - \gamma_o^6 q^6 = 0 \tag{51}$$

which yields the nondimensional phase velocity relation

$$\Omega/q = 1/\alpha - \frac{1}{6\alpha}(1-\beta^4/\alpha^4)q^2 + \frac{1}{9\alpha}\left[\frac{1}{5}(1-\gamma^6/\alpha^6)-(\beta^4/\alpha^4)(1-\beta^4/\alpha^4) - \frac{1}{4}(1-\beta^4/\alpha^4)^2\right]q^4 + \dots \tag{52}$$

This suggests that the fourth-order correction to the long wavelength sound speed is given by

$$\overset{(4)}{C^2} = \frac{1}{81}\left[\frac{1}{5}(1-\gamma^6/\alpha^6)-(\beta^4/\alpha^4)(1-\beta^4/\alpha^4) - \frac{1}{4}(1-\beta^4/\alpha^4)^2\right]^2 \overset{(o)}{C^2} . \tag{53}$$

From Eqns. (48) and (52) we further see that the limiting value of non-dimensional group velocity is

$$(d\Omega/dq)_{q\to o} = 1/\alpha \ , \tag{54}$$

and for long wavelengths

$$d^2\Omega/dq^2 = -\frac{1}{\alpha}(1 - \beta^4/\alpha^4)q + \dots . \tag{55}$$

The non-zero value of curvature for long wavelengths suggests the dispersive character of the waves.

We may conclude this section with the remark that frequency Eqn. (41) is of class C^ω in the neighborhood of a point (q_o,Ω_o), and therefore there exists a Taylor polynomial which approximates this equation to

any given desired accuracy. From the nature of the derivation of our approximate differential equations, and as a consequence of Gelfand-Levitan theorem, it is obvious that our approximate dispersion spectrum given by Eqns. (47) or (52), are actually the Taylor polynomial approximations of Eqn. (41), in the neighborhood of the point (o,o). By increasing the degree of the Taylor polynomial, we can obtain a polynomial approximation with arbitrarily small error. This implies that by increasing the degree of approximate differential equation, we can make the error arbitrarily small. For purposes of comparison, we have shown in Fig. 2, over one Brillouin zone, the exact spectrum, and its various orders of Taylor polynomial approximations as given by Eqns. (38) and (46). The spectrum marked L_4 and L_6, belong to Eqns. $(38)_i$ and $(38)_{ii}$, and those marked H_4 and H_6, belong to Eqns. $(46)_i$ and $(46)_{ii}$. Spectral curve H_8 corresponding to Eqn. (44) or L_8 corresponding to the eighth-order approximation of Eqn. (37) is also shown in the figure. As mentioned earlier in the introduction, the spectrum of the eighth-order theory, matches to a very high degree of accuracy, the exact spectrum, over the entire first Brillouin zone.

Homogenization and Multiple Scale Expansion

We now derive the effective properties of the laminated, periodic composite by homogenization via a multiple scale expansion. Before doing so, we may mention that homogenization can also be carried out by using functional analytic methods [10,11], based on energy considerations and G-convergence of operators [12]. One may also use probabilistic methods [13], or diffusion approximations to random evolutions as in [14]. As a matter of fact these functional analytic and probabilistic methods provide a justification for the multiple scale expansion method, which we now present.

Combining Eqns. (1) and (2) we get the equation

$$\frac{\partial}{\partial y}\left(\eta(y)\,\frac{\partial u}{\partial y}\right) - \rho(y)\,\frac{\partial^2 u}{\partial t^2} = 0, \qquad 0 < y < \infty\ , \qquad t > 0 \tag{56}$$

where we now assume

$$\eta(y) = \eta(y + 2\Delta)\ ; \qquad \rho(y) = \rho(y + 2\Delta)\ . \tag{57}$$

The periodicity of the coefficients of the differential equation with half period $\Delta \equiv (h_1 + h_2)$, reflects the periodic nature of the particular heterogeneous medium under consideration, and we wish to study Eqn. (56) as $\Delta \to 0$.

In order to carry out this limiting process we introduce a new length scale

$$x = y/\varepsilon\ , \qquad \varepsilon > 0\ , \tag{58}$$

and rewrite Eqn. (56) as

$$\frac{\partial}{\partial y}\left(\eta(x)\,\frac{\partial}{\partial y}\,U^{\varepsilon}(x,y;t)\right) - \rho(x)\,\frac{\partial^2}{\partial t^2}\,U^{\varepsilon}(x,y;t)\ . \tag{59}$$

In this equation the superscript ε indicates the dependence of U on ε. We want to study the behavior of Eqn. (59) as $\varepsilon \to 0$. Keeping y fixed and letting ε approach zero, we find that $\eta(x)$ and $\rho(x)$ become more and more oscillatory. Therefore, the study of Eqn. (59) as $\varepsilon \to 0$, does provide us with information about Eqn. (56) as $\Delta \to 0$.

In view of Eqn. (58)

$$\partial/\partial y = \partial/\partial y + \frac{1}{\varepsilon}\,\partial/\partial x\ , \tag{60}$$

and therefore Eqn. (59) can be rewritten as

$$\eta(x)\,\frac{\partial^2 U^{\varepsilon}}{\partial y^2} + \frac{1}{\varepsilon}\left[\eta(x)\,\frac{\partial^2 U^{\varepsilon}}{\partial y \partial x} + \frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial U^{\varepsilon}}{\partial y}\right] + \frac{1}{\varepsilon^2}\,\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial U^{\varepsilon}}{\partial x} = \rho(x)\,\frac{\partial^2 U^{\varepsilon}}{\partial t^2}\ . \tag{61}$$

We now assume that $U^{\varepsilon}(x,y;t)$ has a power series representation

$$U^{\varepsilon}(x,y;t) = \sum_{n=0}^{\infty} \varepsilon^{n}\,\overset{(n)}{U}(x,y;t)\ , \tag{62}$$

where it is assumed that each of the functions $\overset{(n)}{U}$, $n = 0,1,2,\ldots$ are 2Δ-periodic with respect to x ; i.e., $\overset{(n)}{U}(x + 2\Delta,y;t) = \overset{(n)}{U}(x,y;t)$. Substituting this series representation in Eqn. (61) and equating like powers of ε, we get the system of equations

$$\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial \overset{(o)}{U}}{\partial x} = 0 \;,$$

$$\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial \overset{(1)}{U}}{\partial x} = -\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial \overset{(o)}{U}}{\partial y} - \eta(x)\,\frac{\partial^2 \overset{(o)}{U}}{\partial y \partial x} \;, \tag{63}$$

$$\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial \overset{(n+2)}{U}}{\partial x} = -\frac{\partial}{\partial x}\,\eta(x)\,\frac{\partial \overset{(n+1)}{U}}{\partial y} - \eta(x)\,\frac{\partial^2 \overset{(n+1)}{U}}{\partial y \partial x} - \eta(x)\,\frac{\partial^2 \overset{(n)}{U}}{\partial y^2} + \rho(x)\,\frac{\partial^2 \overset{(n)}{U}}{\partial t^2} \;,$$

for $n = 0,1,2,3,\ldots$.

The solution of Eqn. $(63)_1$, which is 2Δ-periodic in x , is easily found to be

$$\overset{(o)}{U} = \overset{(o)}{U}(y,t). \tag{64}$$

Substituting Eqn. (64) in Eqn. $(63)_2$, we find that $\overset{(1)}{U}$ satisfies the equation

$$\frac{\partial}{\partial x}\,\eta(x)\left(\frac{\partial \overset{(1)}{U}}{\partial x} + \frac{\partial \overset{(o)}{U}}{\partial y}\right) = 0 \;. \tag{65}$$

The solution of this equation, which is 2Δ-periodic in x , is given by

$$\overset{(1)}{U} = f_1(x)\,\frac{\partial \overset{(o)}{U}}{\partial y} + g_1(x,y;t), \tag{66}$$

where $f_1(x)$ is the 2Δ-periodic in x solution of the equation

$$\frac{d}{dx}\,\eta(x)\,\frac{df_1}{dx} + \frac{d}{dx}\,\eta(x) = 0 \;, \tag{67}$$

and $g_1(x,y;t)$ is the 2Δ-periodic in x solution of $\frac{\partial}{\partial x}(\eta(x)\partial g_1/\partial x)=0$. Hence, $g_1(x,y;t) = \overset{o}{U}(y,t)$, as can be verified by substituting Eqn. (66) in $(63)_2$.

Equation $(63)_3$ for $n = 0$, has a periodic solution with respect to x iff

$$\int_x^{x+2\Delta} \left[\rho(x)\frac{\partial^2 \overset{(o)}{U}}{\partial t^2} - \eta(x)\frac{\partial^2 \overset{(o)}{U}}{\partial y^2} - \eta(x)\frac{\partial^2 \overset{(1)}{U}}{\partial y \partial x} - \frac{\partial}{\partial x}\eta(x)\frac{\partial \overset{(1)}{U}}{\partial y}\right] dx = 0\ ; \tag{68}$$

since the remaining term when integrated over one period vanishes. Substituting Eqns. (64) and (66) in (68) and simplifying, we get

$$\left(\frac{1}{2\Delta}\int_x^{x+2\Delta} \eta(x)(1 + df_1/dx)dx\right)\frac{\partial^2 \overset{(o)}{U}}{\partial y^2} - \overset{(o)}{\rho}\frac{\partial^2 \overset{(o)}{U}}{\partial t^2} = 0\ , \tag{69}$$

where

$$\overset{(o)}{\rho} \equiv \frac{1}{2\Delta}\int_x^{x+2\Delta} \rho(x)dx \tag{70}$$

is the average density of the material over one period. To find the value of the integral in Eqn. (69), we make use of Eqn. (67). Integrating this equation once we get

$$\eta(x)(1 + f_1'(x)) = \overset{(o)}{\eta}\ , \tag{71}$$

where $\overset{(o)}{\eta}$ is the constant of integration. To determine this constant we integrate Eqn. (71) over one period, and using the periodic property of function $f_1(x)$, we get

$$\frac{1}{\overset{(o)}{\eta}} = \frac{1}{2\Delta}\int_x^{x+2\Delta} (1/\eta(x))dx\ , \tag{72}$$

which is the average of $\eta^{-1}(x)$ over one period. Now substituting Eqn. (71) in Eqn. (69), we get the "homogenized equation"

$$\overset{(o)}{\eta}\,\frac{\partial^2 \overset{(o)}{U}}{\partial y^2} - \overset{(o)}{\rho}\,\frac{\partial^2 \overset{(o)}{U}}{\partial t^2} = 0 , \tag{73}$$

which is the zero-order approximation to Eqn. (56) or (61). This is a one-dimensional wave equation in a homogeneous, isotropic medium whose effective zero-order density and stiffness are given by $\overset{(o)}{\rho}$ and $\overset{(o)}{\eta}$, respectively. In the case of a bilaminate composite, $\overset{(o)}{\rho}$ and $\overset{(o)}{\eta}$ calculated from Eqns. (70) and (72) agree with the results given by Eqns. $(34)_{i,ii}$, respectively. This verification of the results obtained earlier by using Lie-series, provides us with another strong justification for the multiple scale expansion just carried out.

To determine $\overset{(2)}{U}$ we again consider Eqn. $(63)_3$ for n=0. Making use of Eqns. (64),(66) and (73) we get

$$\frac{\partial}{\partial x}\left[\eta(x)\,\frac{\partial \overset{(2)}{U}}{\partial x} + \frac{\partial \overset{(1)}{U}}{\partial y}\right] - \overset{(1)}{\tilde{\rho}}(x)\,\frac{\partial^2 \overset{o}{U}}{\partial t^2} = 0 , \tag{74}$$

where

$$\overset{(1)}{\tilde{\rho}}(x) \equiv \rho(x) - \overset{o}{\rho} , \tag{75}$$

which is the deviation in density from the effective density $\overset{(o)}{\rho}$.

We assume that the solution of Eqn. (74), which is 2Δ-periodic in x is of the form

$$\begin{aligned} \overset{(2)}{U} &= f_2(x)\,\frac{\partial^2 \overset{(o)}{U}}{\partial y^2} + \overset{(1)}{U} , \\ &= f_2(x)\,\frac{\partial^2 \overset{(o)}{U}}{\partial y^2} + f_1(x)\,\frac{\partial \overset{(o)}{U}}{\partial y} + \overset{o}{U} . \end{aligned} \tag{76}$$

Now substituting Eqn. (76) in (74) and making use of Eqns. (71) and (73), we find that for $\overset{(2)}{U}$ to be the 2Δ-periodic solution, $f_2(x)$ must satisfy the equation

$$\frac{d}{dx}[\eta(x)(f_2'(x) + f_1(x))] = (\rho(x) - \overset{(o)}{\rho})C_o^2 , \tag{77}$$

where $C_o^2 \equiv \overset{(o)}{\eta}/\overset{(o)}{\rho}$. Integrating it once we get

$$f_2'(x) = \frac{1}{\eta(x)}(C_o^2 \overset{(1)}{\tilde{\rho}}(x) + \overset{(1)}{\eta}) - f_1(x) , \tag{78}$$

where $\overset{(1)}{\eta}$ is a new constant of integration. Integrating this equation over one period and making use of Eqn. (72), we get

$$\frac{C_o^2}{2\Delta}\int_x^{x+2\Delta} \overset{(1)}{\tilde{\rho}}(x)dx + \overset{(1)}{\eta}/\overset{(o)}{\eta} = \frac{1}{2\Delta}\int_x^{x+2\Delta} f_1(x)dx . \tag{79}$$

However, from Eqn. (71)

$$f_1(x) = \overset{(o)}{\eta}\int_o^x \frac{dx}{\overset{(1)}{\tilde{\eta}}(x)} , \tag{80}$$

where

$$1/\overset{(1)}{\tilde{\eta}}(x) = (1/\eta(x) - 1/\overset{(o)}{\eta}) , \tag{81}$$

which is the deviation in compliance from the effective compliance. Now substituting Eqn. (80) in (79), we find that the effective first order stiffness $\overset{(1)}{\eta}$ is given by

$$\overset{(1)}{\eta}/\overset{(o)}{\eta} = \frac{\overset{(o)}{\eta}}{2\Delta}\left(\int_{x}^{x+2\Delta} dx \int_{o}^{x} \frac{d\xi}{\overset{(1)}{\tilde{\eta}}(\xi)} - \int_{x}^{x+2\Delta} \left(\overset{(1)}{\tilde{\rho}}(x)/\overset{(o)}{\rho}\right)dx\right) . \tag{82}$$

This process can now be continued and effective properties of higher orders can similarly be determined from Eqns. $(63)_3$, $n = 2,3,\ldots$.

We now note that when

$$\begin{aligned}
\overset{(1)}{U} &= f_1(x) \frac{\partial \overset{(o)}{U}}{\partial y} + \overset{o}{U} , \\
\overset{(2)}{U} &= f_2(x) \frac{\partial^2 \overset{(o)}{U}}{\partial y^2} + f_1(x) \frac{\partial \overset{(o)}{U}}{\partial y} + \overset{o}{U} , \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
\overset{(n)}{U} &= f_n(x) \frac{\partial^n \overset{(o)}{U}}{\partial y^n} + f_{n-1}(x) \frac{\partial^{n-1} \overset{(o)}{U}}{\partial y^{n-1}} + \ldots + f_1(x) \frac{\partial \overset{(o)}{U}}{\partial y} + \overset{(o)}{U} ,
\end{aligned} \tag{83}$$

the power series representation for displacement field takes the form

$$U^{\varepsilon}(x,y;t) = (1 + \varepsilon f_1 \frac{\partial}{\partial y} + \varepsilon^2 f_2 \frac{\partial^2}{\partial y^2} + \ldots)V(y,t) , \tag{84}$$

where

$$V(y,t) = \overset{o}{U}(1 + \varepsilon + \varepsilon^2 + \ldots) = \frac{\overset{o}{U}}{(1-\varepsilon)} , \tag{85}$$

and satisfies the "homogenized equation" (73).

It can be shown (cf. [14]) that U^{ε} converges weakly to $\overset{o}{U}$ in

$L^2((0,\infty)$, $H_o^1(0,\infty))$ and that $U^\varepsilon(x,y;t) - (1 + \varepsilon f_1(x) \frac{\partial}{\partial y})V(y,t)$ converges strongly to 0 in the same space, (cf. [10]). Here $H_o^1(0,\infty) = \{f(y) : f$ and $\frac{df}{dy} \varepsilon L^2(0,\infty)$, $f(o) = f(\infty) = 0\}$.

General Remarks

The general solution to Eqns. (1) and (2) is a quasiperiodic Floquet wave of the form

$$U(x,t) = v(x) \exp[i(qx - \Omega t)] \quad , \tag{86}$$

where the amplitude function v is 2Δ-periodic in x, i.e., $v(x + 2\Delta) = v(x)$.

As was shown in part one, the Lie-series approximation amounts to assuming that the frequency Ω can be expanded in the power series (43). Thus, the Lie-series approximation is a long wavelength approximation.

The approximation obtained by homogenization in part two corresponds to an expansion of the amplitude function as

$$v(x) = 1 + iq\, v_1 + (iq)^2 v_2 + \ldots \quad . \tag{87}$$

Note however that the wave number q in this expansion corresponds to that of a wave propagating in a homogeneous medium, the properties of which are the effective properties. This approximation, in which q is of the same order as Ω but differs in order from the q entering in the Lie-series approximation, is therefore a low frequency approximation.

The approximation suggested by Kohn in [4] is one in which both frequency Ω and amplitude v are expanded in power series in wave number q . It is thus a long wavelength-low frequency approximation

as pointed out by Kohn himself.

The merits of each approximation, which will dictate the choice of one approximation over the others, depend upon the region of interest in the (Ω,q) space as well as the nature of the problem under investigation (e.g., transient versus steady-state problems).

The way in which microstructure is taken into account in the homogenization expansion is worth a few comments. Indeed, f_1 introduces a correction for variable stiffness only, (cf. Eqn. (79)); a correction for variable density first occurs through f_2 , (cf. Eqn. (78)). Both stiffness and density enter into the higher order correction terms f_n, $n>2$, the detailed nature of which can easily be obtained in any particular situation.

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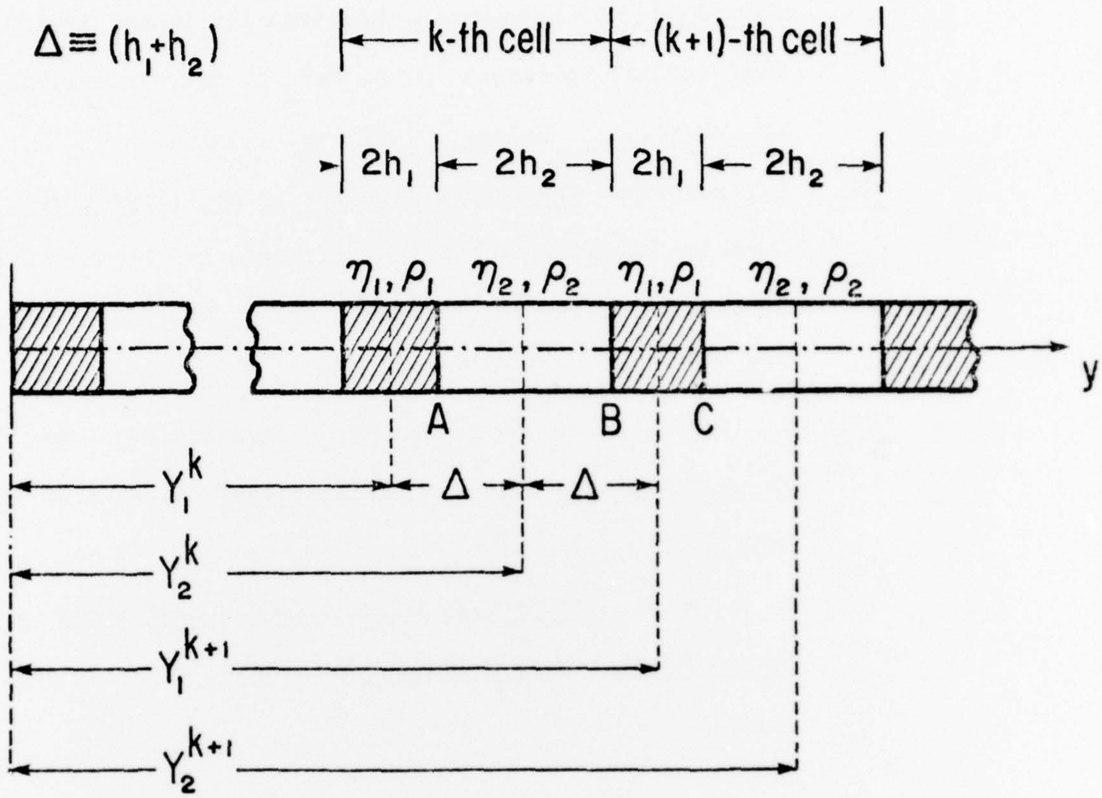

Fig. 1 One-dimensional laminated composite with periodic structure. Half-period $\Delta \equiv (h_1+h_2)$.

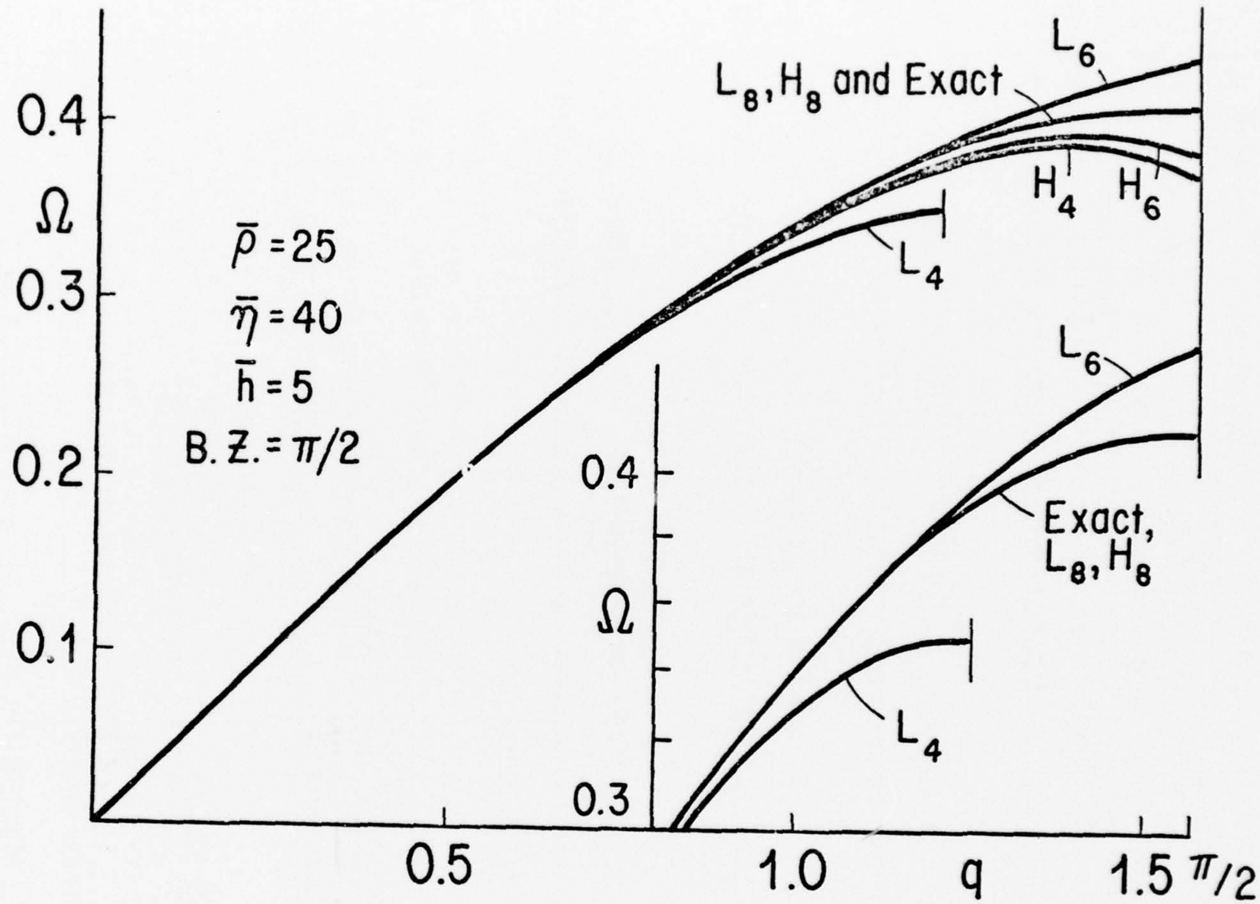

Fig. 2. Dispersion spectrum of fourth, sixth and eighth order equations over one Brillouin zone of width $\pi/2$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we develop approximate theories for longitudinal wave propagation in one-dimensional laminated composite with periodic structure. We first show how the method of Lie-series can be used to develop higher order equations with constant coefficients. In the second part of the paper it is shown that equivalent results can also be obtained by using the ideas of multiple scale expansion and homogenization.